Gauge theory at Bocconi
Lecture 1: Abelian aspects

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1 What is gauge theory?

There are many good answers and some (biased) answers are:

- A theory for describing physical phenomena independent of local measurement choices.
- The theory of (special) connections up to infinite-dimensional symmetries, that is, it is the geometry of infinite-dimensional group actions.
- A natural intersection of complex geometry (Kähler, hyperKähler), elliptic PDE (Dirac operators), algebraic geometry (moduli spaces), and topology (Index theorem, equivariant cohomology).

2 Maxwell with forms

Consider $\mathbb{R}^4$ equipped with the coordinates (t, x^1, x^2, x^3) with the Minkowski metric

$$\eta = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Note that later we will switch to a Euclidean signature.

The first instance of gauge theory is already given by the field theory of electromagnetism, described by **Maxwell's equations**

$$\begin{aligned} \nabla \cdot \overrightarrow{E} &= \rho, \\ \nabla \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} &= 0, \\ \nabla \cdot \overrightarrow{B} &= 0, \\ \nabla \times \overrightarrow{B} - \frac{\partial \overrightarrow{E}}{\partial t} &= \overrightarrow{J}. \end{aligned}$$

where $\overrightarrow{E}, \overrightarrow{B}, \overrightarrow{J} : \mathbb{R}^4 \to \mathbb{R}^3$ and $\rho : \mathbb{R}^4 \to \mathbb{R}$. The stated equations are for units where the speed of light is set to 1.

Let us reinterpret $\overrightarrow{E}$ and $\overrightarrow{B}$ as differential forms. In this setting we consider

$$\begin{aligned} &E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3 = E_i\, dx^i, \\ &B = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 = B_i\, dx^j \wedge dx^k \quad \text{with } (ijk) \text{ cyclic.} \end{aligned}$$

Define the **Faraday tensor**

$$F = E \wedge dt + B = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

and

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix}.$$

Define also

$$J = -\rho\, dt + J_1\, dx^1 + J_2\, dx^2 + J_3\, dx^3.$$

Proposition 2.1. *Maxwell's equations in terms of differential forms are*

$$dF = 0,$$
$$d \star F = \star J,$$

where $\star$ *denotes the Hodge star operator of* $(\mathbb{R}^4, \eta)$*.*

Remark. The important part of the Hodge star operator for us is described by

$$\star(dt \wedge dx^1) = -dx^2 \wedge dx^3,$$
$$\star(dt \wedge dx^2) = dx^1 \wedge dx^3,$$
$$\star(dt \wedge dx^3) = -dx^1 \wedge dx^2,$$

and $\star^2 = -1$ on 2-forms.

Corollary 2.2. *Let* (M, g) *be a compact Riemannian manifold. Consider the set*

$$\mathcal{M} = \{F \in \Omega^2(M) \mid dF = 0 = d^* F\}.$$

Then

$$\mathcal{M} \cong \mathcal{H}^2(M, \mathbb{R}).$$

2.1 Gauge potentials

Notation 2.3. We now assume that F is $i\mathbb{R}$-valued instead of just being $\mathbb{R}$-valued. This convention is simply adopted because the Lie algebra of

$$\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$$

is usually identified with $\mathfrak{u}(1) = i\mathbb{R}$.

Locally, by the Poincaré lemma, we know that there is always an $A \in \Omega^1(U, i\mathbb{R})$ such that $dA = F$, i.e. we have a **potential** (connection) that we can use to describe F. Note that $A \mapsto A - i\, d\alpha$ is an equally good choice as

$$d(A - i\, d\alpha) = dA - i\, d^2\alpha = F.$$

This non-uniqueness can be implemented in the following way. We know that on U the equivalence

$$A \sim A - i\, d\chi$$

for any choice of $\chi \in \mathscr{C}^\infty(U, \mathbb{R})$ is valid. In this case we can define an action of the **gauge group**

$$\mathcal{G} = \mathscr{C}^\infty(U, S^1)$$

on A via

$$g.A = gAg^{-1} - dg\, g^{-1} = A - i\, d\chi$$

for $g = e^{i\chi}$. The fact that $gAg^{-1} = A$ is the crucial property of abelian gauge theory and is an enormous simplification.

Note that $g : M \to S^1$ is more general than $g = e^{i\chi}$, and this form can only be achieved locally.

Now, the potentials are just local objects. How do we get an object defined everywhere on M? For this consider an open cover $\{U_\alpha\}$ of M such that $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is contractible and repeat the previous step to obtain $dA_\alpha = F|_{U_\alpha}$. Now on the intersections we have

$$dA_\alpha|_{U_{\alpha\beta}} = F|_{U_{\alpha\beta}} = dA_\beta|_{U_{\alpha\beta}}$$

and hence $d(A_\alpha - A_\beta) = 0$. On $U_{\alpha\beta}$ we can again apply the Poincaré lemma to obtain $\chi_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}$ such that

$$i\, d\chi_{\alpha\beta} = A_\alpha - A_\beta.$$

Now, we can define $g_{\alpha\beta} = e^{i\chi_{\alpha\beta}}$ and notice that

$$g_{\alpha\beta}.A_\alpha = A_\alpha - i\, d\chi_{\alpha\beta} = A_\beta$$

on $U_{\alpha\beta}$. Now, for the $g_{\alpha\beta}$ to define a line bundle, they need to satisfy the cocycle conditions

$$\begin{aligned} g_{\alpha\alpha} &= \mathrm{Id} \quad \text{on } U_\alpha, \\ g_{\alpha\beta}g_{\beta\alpha} &= \mathrm{Id} \quad \text{on } U_{\alpha\beta}, \\ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} &= \mathrm{Id} \quad \text{on } U_{\alpha\beta\gamma}, \end{aligned}$$

and these translate to

$$\chi_{\alpha\beta} + \chi_{\beta\gamma} + \chi_{\gamma\alpha} = 0 + 2\pi\mathbb{Z}.$$

Proposition 2.4. *The cocycles $g_{\alpha\beta}$ define a line bundle if and only if*

$$\left[\frac{1}{2\pi i}F\right] \in H^2(M, \mathbb{Z}).$$

Remark. This is the same condition as the prequantisation condition for a symplectic manifold (M, ω).

2.2 Weyl's gauge principle (also of Fock and London)

In Hermann Weyl's quest to understand fundamental physics from basic principles, he discovered in 1929 the gauge principle in physics. (He even coined the name "gauge theory," which is *Eichtheorie* in German.) The idea is a simple observation from quantum mechanics.

As a state of a quantum system is not just the vector $\psi : M \to \mathbb{C}$ but the complex line it defines in its Hilbert space $\mathcal{H}$, the phase-multiplied state $e^{i\theta}\psi$ for $\theta \in \mathbb{R}$ is an equivalent description of the same state. Thus there is an S^1 symmetry present in quantum mechanics.

Hermann Weyl's gauge principle stems from the simple idea of promoting S^1 to a local symmetry, i.e. to multiplication by $e^{i\chi}$ with a function $\chi : M \to \mathbb{R}$. The computation

$$d(e^{i\chi}\psi) = e^{i\chi}\,(d + i\, d\chi)\,\psi$$

shows that the idea of differentiation needs to be changed.
The idea of Weyl was the following. If we define

$$\nabla_A := d + A$$

for $A \in \Omega^1(M, i\mathbb{R})$ with the transformation behaviour

$$g.A = A - dg\, g^{-1}$$

for $g \in \mathcal{G}$, then

$$\begin{aligned}\nabla_{g.A}(g\psi) &= (d + A - dg\, g^{-1})g\psi \\ &= dg\,\psi + g\,d\psi + gA\,\psi - dg\,\psi \\ &= g(d+A)\psi \\ &= g\nabla_A\psi.\end{aligned}$$

Thus, if we want to have the gauge symmetries of our theory, then the fundamental object is actually the pair (A, ψ) transforming as

$$g.(A, \psi) = (g.A, g\psi)$$

for $g \in \mathcal{G}$.

Example 2.5. Consider on $\mathbb{R}$ the equation

$$\frac{d}{dt}\psi = 0$$

for $\psi : \mathbb{R} \to \mathbb{C}$. A gauge-invariant upgrade of this would be

$$\left(\frac{d}{dt} + ia(t)\right)\psi = 0.$$

While the first equation has constants as solutions, the second equation has the solution

$$\psi(t) = C\exp\left(-i\int_0^t a(\tau)\,d\tau\right).$$

Remark. By the previous discussions, we saw that Maxwell's equations are really meant to be written in terms of a potential A for the Faraday tensor $F = dA$. In this case there is also a nice variational formulation in terms of

$$S[A] = \int_{\mathbb{R}^4} -\frac{1}{2}F \wedge \star F + A \wedge \star J.$$

Note that in Riemannian signature with $J = 0$ this is just

$$E[A] = \frac{1}{2}\int_M F \wedge \star F = \frac{1}{2}\|F\|^2_{L^2}.$$

Remark. One might ask whether there is any physics hidden in the gauge potential A or whether it is just a mathematical tool. In 1959 Aharonov and Bohm published their result constructing an experiment that can decide this. They took an infinite solenoid and from that they derived that indeed the potential A should have physical significance.

2.3 A simple moduli space

As a further example, we consider a moduli space problem on a compact Riemannian manifold (M, g). Namely, we want to understand

$$\mathcal{M}_{dR} = \{A \in \Omega^1(M, i\mathbb{R}) \mid F_A = 0\}/\mathcal{G}$$

with the gauge group $\mathcal{G} = \mathscr{C}^\infty(M, S^1)$ acting via

$$g.A = A - dg\, g^{-1}.$$

Understanding such problems can be put into a schematic.

- **Step 1:** Linearise the equations and the gauge action

$$D_0 \xrightarrow{d_0} D_1 \xrightarrow{d_1} D_2$$

- **Step 2:** Prove Fredholm/elliptic properties.
- **Step 3:** Conclude the manifold (or more general) structure.

Step 1. Consider $\mathcal{F} : \Omega^1(M, i\mathbb{R}) \to \Omega^2(M, i\mathbb{R})$ given by $\mathcal{F}(A) = dA$. The moduli space is

$$\mathcal{M}_{dR} = \mathcal{F}^{-1}(0)/\mathcal{G}.$$

The derivative of $\mathcal{F}$ at A in the direction $a \in \Omega^1(M, i\mathbb{R})$ is

$$D_A\mathcal{F}(a) = \frac{d}{dt}\bigg|_{t=0} \mathcal{F}(A + ta) = da.$$

The linearisation of the gauge action is

$$\frac{d}{dt}\bigg|_{t=0} (e^{it\chi}.A) = \frac{d}{dt}\bigg|_{t=0} (A - it\, d\chi) = -i\, d\chi.$$

The linearisation is essentially

$$\Omega^0(M, \mathbb{R}) \xrightarrow{d_0} \Omega^1(M, i\mathbb{R}) \xrightarrow{d_1} \Omega^2(M, i\mathbb{R})$$

with $d_0(\chi) = -i\, d\chi$, $d_1(a) = da$. By completing the vector spaces to the appropriate Sobolev spaces we are finished with Step 1. Essentially what we are considering here is

$$T^{\mathrm{vir}}_{[A]}\mathcal{M}_{dR} = \frac{\ker d_1}{\operatorname{Im}\, d_0}.$$

Step 2. Now we can consider the operator

$$D := d_1 \oplus d_0^* : \Omega^1(M, i\mathbb{R}) \to \Omega^2(M, i\mathbb{R}) \oplus \Omega^0(M, \mathbb{R})$$

which is

$$D(a) = (da, id^*a),$$

and in this case

$$T^{\mathrm{vir}}_{[A]}\mathcal{M}_{dR} = \ker D$$

and consists of harmonic forms, i.e.

$$da = d^*a = 0.$$

Thus by Hodge theory

$$T^{\mathrm{vir}}\mathcal{M}_{dR} \cong \mathcal{H}^1(M).$$

This already tells us that the expected dimension is $b_1(M)$.

Step 3. Actually in our discussion the linearisation was artificial, as $\mathcal{F}$ is already linear. We have

$$\ker \mathcal{F} = \Omega^1_{\mathrm{cl}}(M, i\mathbb{R})$$

and the gauge group acts via

$$g.A = A - dg\, g^{-1}.$$

Now $dg\, g^{-1}$ is a closed form that has periods in $2\pi i\,\mathbb{Z}$. This quickly leads to the description

$$\mathcal{M}_{dR} = \frac{\Omega^1_{\mathrm{cl}}(M, i\mathbb{R})}{\mathcal{G}} \cong \frac{H^1(M, i\mathbb{R})}{2\pi i H^1(M, \mathbb{Z})} \cong \mathbb{T}^{b_1(M)}.$$

This is the most basic case of the **Riemann–Hilbert correspondence**.

3 Next lecture

The next lecture will cover the non-abelian analogue of today's lecture, where the structure group G changes from $\mathrm{U}(1)$ to a general (possibly non-abelian) Lie group G.